17 Suppose that $V_1, \dots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

The inequality above is an equality if and only if $V_1 + \cdots + V_m$ is a direct sum, as will be shown in 3.94.

Suppose that $V_1, \dots, V_m$ are finite dimensional subspace of V.

Base case | ($m=2$)

NTS: $\dim(V_1+V_2) \leq \dim(V_1)+\dim(V_2)$

Let $d_1 = \dim(V_1)$ and
$d_2 = \dim(V_2)$

~~Since There exist basis~~

Furthur, $V_1 = \operatorname{span}(\gamma_1^1, \dots, \gamma_{d_1}^1)$

$V_2 = \operatorname{span}(\gamma_1^2, \dots, \gamma_{d_2}^2)$

Let $u_1 \in V_1$ and $u_2 \in V_2$

Then $u_1 = a_1\gamma_1^1 + \dots + a_{d_1}\gamma_{d_1}^1$ forsome $a_1, \dots, a_{d_1} \in \mathbb{F}$

$u_2 = b_1\gamma_1^2 + \dots + b_{d_2}\gamma_{d_2}^2$ forsome $b_1, \dots, b_{d_2} \in \mathbb{F}$

$u_1 + u_2 \in \operatorname{span}(\gamma_1^1, \gamma_2^1, \dots, \gamma_{d_1}^1, \gamma_1^2, \dots, \gamma_{d_2}^2)$

Thus, $V_1 + V_2 \subseteq \operatorname{span}(\gamma_1^1, \dots, \gamma_{d_1}^1, \gamma_1^2, \dots, \gamma_{d_2}^2)$

Thus, $\dim(U_1+U_2) \leq d_1 + d_2 = \dim(V_1) + \dim(V_2)$

Induction hypothesis

Assume that for $m \geq 3$,

$\dim(V_1 + \cdots + V_m) \leq \dim(V_1) + \cdots + \dim(V_m)$

Now $m+1$ Case

$\dim(V_1 + \cdots + V_m + V_{m+1}) \leq \dim(V_1 + \cdots + V_m) + \dim(V_{m+1})$

(By base case $m=2$)

$\dim(V_1 + \cdots + V_{m+1}) \leq \dim(V_1) + \cdots + \dim(V_m) + \dim(V_{m+1})$

(by induction hypothesis)

Therefore, by ~~induction~~ mathematical ~~dim(V1+~~ induction hypothesis,

$\dim(V_1) + \cdots + \dim(V_m) \geq \dim(V_1 + \cdots + V_m)$

for all $m \in \mathbb{Z}^+$

■